

# On the Covariance of the Charge Form Factor in the Transition Radiation Energy Spectrum of a Beam at Normal Incidence onto a Metallic Screen

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## Abstract

A charge-density-like covariance is expected to characterize the transition radiation energy spectrum of a  $N$  electron bunch as far as the charge form factor is intended to account for bunch-density effects in the radiation emission. The beam charge passing from a single electron to a high density electron bunch, the covariance of the transition radiation energy is expected to evolve from a charge-point-like to a charge-density-like one. Besides covariance, the radiation energy spectrum is expected to conform to the temporal causality principle: the  $N$  single electron amplitudes composing the radiation field are expected to propagate from the metallic screen with relative emission phases causally correlated with the temporal sequence of the  $N$  particle collisions onto the metallic screen. In the present paper, the case of a  $N$  electron bunch hitting at a normal angle of incidence a flat metallic surface with arbitrary size and shape will be considered. For such an experimental situation, the distribution function of the  $N$  electron longitudinal coordinates rules the temporal causality constraint into the transition radiation energy spectrum. The covariance feature of the transition radiation energy spectrum deals instead with the Lorentz invariance of the projection of the  $N$  electron spatial density in the transverse plane with respect to the direction of motion of the  $N$  electron beam. Because of the invariance of the  $N$  electron transverse density under a Lorentz transformation with respect to the direction of motion of the electron beam, the  $N$  single electron radiation amplitudes composing the radiation field show a covariant dependence on the distribution function

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of the  $N$  electron transverse coordinates, the relative emission phases of the  $N$  single electron radiation amplitudes being indeed only a function of the  $N$  electron longitudinal coordinates because of the temporal causality constraint. As a consequence of the temporal causality and the covariance, both the temporal coherent and incoherent components of the radiation energy spectrum bear the covariant imprinting of the distribution function of the  $N$  electron transverse coordinates as in the following argued.

*Keywords:* Virtual Quanta, Coherence, Fourier Transform, Collective Effects

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## 1. Introduction

A relativistic charge in a rectilinear and uniform motion can originate an electromagnetic radiative mechanism crossing the dielectric interface between two different media. The fast dipolar oscillation of the polarization charge, induced on the dielectric interface by the relativistic charge, generates indeed an instantaneous, broad spectral band, radially polarized and highly directional radiation emission, the so called transition radiation [1, 2, 3, 4, 5, 6, 7, 8]. A photon pulse propagates backward and forward from the dielectric interface according to a double conical spatial distribution whose angular aperture scales down with the Lorentz  $\gamma$  factor of the charge. The higher and steeper the discontinuity of the dielectric constant across the interface, the more intense the radiation emission.

Thanks to the relativistic origin, transition radiation can be fruitfully exploited for energy detection or mass identification of high energy particles [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Transition radiation based diagnostics is commonly used in a particle accelerator to monitor the transverse [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37] or the longitudinal profile of a charged beam [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55]. Transition radiators, typically made of a thin metallic foil or a thin polished Aluminium coating on dielectric substrates, respond as an ideal conductor over a wide spectral range, from the long wavelength to far beyond the visible optical region [56]. This experimental situation is supposed in the present work. In particular, the case of a  $N$  electron bunch colliding at a normal angle of incidence onto a flat ideal conductor surface will be considered in the following.

26 An ideal conductor surface being assimilable to a double layer of charge,  
 27 transition radiation emission can be thus schematized as the result of the  
 28 interaction of an incident charge with the conduction electrons of the ideal  
 29 conductor surface. As the relativistic charge approaches the metallic surface,  
 30 the charge induced conduction electrons freely move in the transverse plane  
 31 to maintain the metallic surface equipotential. The sudden and fast dipolar  
 32 oscillation of the double layer of charge is thus responsible for the emission  
 33 of the radiation pulse as confirmed by the emission of backward radiation.

34 In the relativistic limit, the electromagnetic field travelling with a rel-  
 35 ativistic electron can be assimilated to a transverse electromagnetic wave,  
 36 the so called *virtual quanta* field [7, 57]. In the case of a single relativistic  
 37 electron, the harmonic component of the electric field at a given wavelength  
 38  $\lambda$  extends indeed in the transverse plane over an efficacious range in the or-  
 39 der of  $\gamma\lambda/2\pi$ , the ratio of the longitudinal to the transverse component being  
 40  $1/\gamma^2$ . The *virtual quanta* field showing the nature of a quasi-plane wave front,  
 41 the radiation emission can be formally described as the result of the wave  
 42 propagation of the *virtual quanta* scattered by the metallic surface according  
 43 to the Huygens-Fresnel principle [58]. Under the far field approximation [7],  
 44 the harmonic components of the transition radiation field at the observation  
 45 point can be calculated as the Fourier transform of the *virtual quanta* field  
 46 with respect to the spatial coordinates of the radiator surface. The transition  
 47 radiation energy spectrum can be finally obtained as the flux of the Poynting  
 48 vector.

49 A single and individually radiating electron and a high density electron  
 50 beam are in general different from the point of view of the features of the  
 51 spectral and the angular distribution of the emitted radiation. Compared to  
 52 the case of a single and individually radiating electron, bunch-density effects  
 53 may strongly affect the electromagnetic radiative mechanism by an electron  
 54 beam. For a fixed energy of the beam, the number and the angular distribu-  
 55 tion of the photons radiated at a given wavelength may indeed change as a  
 56 function of the density of the electron beam. In the formula of the radiation  
 57 energy spectrum, the charge form factor accounts for bunch collective effects.  
 58 This is, in principle, defined as the square module of the Fourier transform  
 59 of the distribution function of the particle density of the charged beam.

60 Aim of the present paper is to verify how well such a theoretical definition  
 61 of the charge form factor fits into a covariant formulation of the radiation en-  
 62 ergy spectrum or if, instead, under the covariance and the temporal causality  
 63 constraints, the radiation energy spectrum shows a dependence on the charge

density that goes beyond the above mentioned formal definition of the charge form factor. In fact, the transverse density of the electron beam is a relativistic invariant under a Lorentz transformation with respect to the direction of motion of the beam. The invariance of the projection of the electron beam density onto the transverse plane with respect to the direction of motion is expected to leave, on the *virtual quanta* field and, consequently, on the radiation field, a covariant mark whose observability, for a given harmonic component of the radiation field, only depends on the transverse dimension of the electron beam compared to the transverse component of the wave-vector which is itself a Lorentz invariant with respect to the direction of motion of the beam. In the following, the charge-density-like covariance of the transition radiation energy spectrum by an electron beam will be investigated. First, the formal steps leading from the electromagnetic field of a  $N$  electron bunch to the transition radiation energy spectrum will be analyzed with regard to the temporal causality constraint. Finally, the covariance of the given formula of the electromagnetic field of the  $N$  electron bunch will be checked by verifying that, under a Lorentz transformation from the laboratory to the rest reference frame, the expected formula of the electric field of a charge distribution at rest is obtained.

## 2. Virtual quanta field and transition radiation energy spectrum of a $N$ electron beam

In the following,  $N$  electrons in motion with a common rectilinear and uniform velocity  $\vec{w}$  along the  $z$  axis of the laboratory reference frame are supposed to strike, at a normal angle of incidence, onto a flat ideal conductor surface  $S$ . The radiator, being in vacuum and placed in the plane  $z = 0$  of the laboratory reference frame, is supposed to have an arbitrary size and shape. Compared to the synchrotron motion of a single electron in the bunch, the time scale of the emission of the radiation pulse by the entire electron bunch can be considered instantaneous. Consequently, in modeling the radiation emission from the electron bunch, collisions between electrons in the bunch can be neglected and the  $N$  electron bunch can be thus described in terms of a “frozen-in-time” spatial distribution function, i.e., a spatial distribution function that is invariant under a time-space translation in the laboratory reference frame (see also Appendix B). With reference to such an experimental context, the temporal sequence of the  $N$  electron collisions onto the metallic screen is only ruled by the distribution function of the  $N$

electron longitudinal coordinates  $z_{0j}$  ( $j = 1, \dots, N$ ) which, for the sake of ease, are taken at the time  $t = 0$  when the center of mass of the electron bunch is supposed to strike the metallic surface.

On the radiator surface  $S$  ( $z = 0$ ), the harmonic components of the transverse electric field of both the relativistic charge ( $E_{x,y}^{vq}$ ) and the charge induced conduction electrons ( $E_{x,y}$ ) satisfy the following boundary constraint:

$$E_{x,y}^{vq}(x, y, z = 0, \omega) + E_{x,y}(x, y, z = 0, \omega) = 0. \quad (1)$$

The boundary condition above rules the dipolar oscillation of the conduction electrons, which is induced on the ideal conductor surface by the incident relativistic charge, and explains why a radiation field needs to fly away from the boundary surface in order to maintain it equipotential. The transition radiation emission can be thus interpreted as the result of the scattering of an electromagnetic wave from a metallic surface and described, on the basis of the Huygens-Fresnel principle, as the wave propagation of the transverse component ( $E_{x,y}^{vq}$ ) of the *virtual quanta* field [7, 57] from the surface  $S$  ( $z = 0$ ) to the observation point. The transition radiation field ( $E_{x,y}^{tr}$ ) can be calculated by means of the Helmholtz-Kirchhoff integral theorem [57, 58]. Under the far-field approximation, this integral can be finally expressed as the Fourier transform of the harmonic component of the *virtual quanta* field with respect to the spatial coordinates  $\vec{\rho} = (x, y)$  of the radiator surface  $S$  [7, 57, 58]:

$$E_{x,y}^{tr}(\vec{\kappa}, \omega) = \frac{k}{2\pi R} \int_S d\vec{\rho} E_{x,y}^{vq}(\vec{\rho}, \omega) e^{-i\vec{\kappa} \cdot \vec{\rho}} \quad (2)$$

where  $k = \omega/c$  is the wave-number,  $\vec{\kappa} = (k_x, k_y)$  is the transverse component of the wave-vector and  $R$  is the distance from the radiator surface to the observation point that, in the present context, is supposed to be on the  $z$ -axis of the laboratory reference frame.

In the case of  $N$  electrons in a rectilinear and uniform motion with a common velocity  $\vec{w} = (0, 0, w)$ , the harmonic component of the *virtual quanta* electric field at the radiator surface  $S$  ( $z = 0$ ) reads, see [60, 61, 62, 63, 64, 65] and also Appendix D:

$$E_{x,y}^{vq}(x, y, z = 0, \omega) = \frac{ie}{w\pi} \sum_{j=1}^N e^{-i(\omega/w)z_{0j}} \int d\vec{\tau} e^{i\vec{\tau} \cdot \vec{\rho}} \frac{\tau_{x,y} e^{-i\vec{\tau} \cdot \vec{\rho}_{0j}}}{\tau^2 + \alpha^2}, \quad (3)$$

where  $\alpha = \frac{\omega}{w\gamma}$ , while  $\vec{\rho}_{0j} = (x_{0j}, y_{0j})$  and  $z_{0j}$  ( $j = 1, \dots, N$ ) are, respectively, the transverse and the longitudinal coordinates of the electrons in the laboratory reference frame at the time  $t = 0$  when the center of mass of the electron bunch is supposed to strike the metallic surface. Taking into account Eqs.(2,3), the transition radiation electric field by a  $N$  electron bunch reads:

$$E_{x,y}^{tr}(\vec{\kappa}, \omega) = \sum_{j=1}^N H_{x,y}(\vec{\kappa}, \omega, \vec{\rho}_{0j}) e^{-i(\omega/w)z_{0j}} \quad (4)$$

where

$$H_{\mu}(\vec{\kappa}, \omega, \vec{\rho}_{0j}) = H_{\mu,j} = \frac{iek}{2\pi^2 R w} \int_S d\vec{\rho} \int d\vec{\tau} \frac{\tau_{\mu} e^{-i\vec{\tau} \cdot \vec{\rho}_{0j}}}{\tau^2 + \alpha^2} e^{i(\vec{\tau} - \vec{\kappa}) \cdot \vec{\rho}} \quad (5)$$

with  $\mu = x, y$ .

With reference to Eqs.(4,5), the transition radiation energy spectrum by a  $N$  electron beam can be finally calculated as the flux of the Poynting vector:

$$\begin{aligned} \frac{d^2 I}{d\Omega d\omega} &= \frac{cR^2}{4\pi^2} \left( |E_x^{tr}(k_x, k_y, \omega)|^2 + |E_y^{tr}(k_x, k_y, \omega)|^2 \right) = \\ &= \frac{cR^2}{4\pi^2} \sum_{\mu=x,y} \left( \sum_{j=1}^N |H_{\mu,j}|^2 + \sum_{j,l(j \neq l)=1}^N e^{-i(\omega/w)(z_{0j}-z_{0l})} H_{\mu,j} H_{\mu,l}^* \right) \end{aligned} \quad (6)$$

where in previous equation the radiator surface  $S$  has, in general, an arbitrary shape and size (either infinite  $S = \infty$  or finite  $S < \infty$ ).

The *virtual quanta* field - see Eq.(3) - shows a field structure as a train of  $N$  travelling transverse waves hitting the metallic surface with a relative phase delay only dependent on the difference between the longitudinal coordinates of the  $N$  electrons  $z_{0j}$  ( $j = 1, \dots, N$ ). In the same way, the radiation field - Eqs.(4,5) - results from the linear addition of  $N$  single particle field amplitudes  $H_{\mu,j}$  whose relative emission phases -  $e^{-i(\omega/w)z_{0j}}$ , ( $j = 1, \dots, N$ ) - from the metallic surface causally depend on the temporal sequence of the  $N$  electron collisions onto the metallic screen.

The temporal causal structure characterizing the transition radiation field of the  $N$  electron bunch - Eqs.(4,5) - is also a feature of the radiation energy spectrum - see Eq.(6) - where the  $N$  single particle field amplitudes  $H_{\mu,j}$  interfere indeed each other via a relative phase factor only dependent on the

relative difference of the electron longitudinal coordinates  $z_{0j}$  ( $j = 1, \dots, N$ ). In fact, (1) the radiator surface  $S$  being in principle arbitrary and (2) the formal expression of the transition radiation energy spectrum being expected to be invariant whether the radiator surface  $S$  is finite ( $S < \infty$ ) or infinite ( $S = \infty$ ) and whatever is its shape, the  $N$  single electron radiation field amplitudes  $H_{\mu,j}$  ( $j = 1, \dots, N$ ) - Eqs.(4,5) - can be formulated in terms of an implicit integral form and treated as a sort of special function whose numerical value can be only formally and implicitly stated in view of the general formulation of the radiation energy spectrum, Eq.(6). Therefore, thanks to the arbitrariness of the radiator surface  $S$  and to the consequent implicit integral formulation of the  $N$  single electron radiation field amplitudes, the formula of the transition radiation energy spectrum - Eq.(6) - explicitly shows the temporal causal feature: the interference between the different single electron field amplitudes  $H_{\mu,j}$  is indeed ruled by the distribution function of the electron longitudinal coordinates  $z_{0j}$  ( $j = 1, \dots, N$ ) via the phase factor  $e^{-i(\omega/w)(z_{0j}-z_{0l})}$  ( $l \neq j$ ).

About the electron transverse coordinates  $\vec{\rho}_{0j} = (x_{0j}, y_{0j})$  ( $j = 1, \dots, N$ ), they play the role to contribute, as a function of the electron displacement from the  $z$ -axis where the radiation field is supposed to be observed, not only to the global phase factor of the single electron field amplitude  $H_{\mu,j}$  ( $j = 1, \dots, N$ ) at the observation point but also to the single electron field amplitude itself. It can be indeed demonstrated [59] that, in the case of a round screen with a finite radius, the integral representing the  $N$  single electron amplitudes composing the radiation field - Eqs.(4,5) - can be solved showing the structure of the product of a real amplitude and a phase factor both depending on the electron transverse coordinates:

$$H_{\mu,j} = H_{\mu}(\vec{\kappa}, \omega, \vec{\rho}_{0j}) = A_{\mu}(\vec{\kappa}, \omega, \vec{\rho}_{0j}) e^{-i\vec{\kappa} \cdot \vec{\rho}_{0j}}. \quad (7)$$

On the basis of Eq.(7), the formula of the radiation energy spectrum given in Eq.(6) shows the well known dependence on the three-dimensional form factor as well as an intrinsic dependence of the  $N$  single electron radiation field amplitudes on the electron transverse coordinates. The transverse density of the  $N$  electrons is indeed an invariant under a Lorentz transformation with respect to the direction of motion of the beam. The transverse coordinates of the  $N$  electrons are thus expected: (1) to contribute to the relative phase delay of the  $N$  electron field amplitudes at the observation point as a function of the transverse displacement of the  $N$  electrons with respect to the beam axis where the radiation field is supposed to be observed; (2) because

188 of the Lorentz invariance of the transverse projection of the  $N$  electron den-  
 189 sity, to leave a covariant mark on the  $N$  electron radiation field amplitudes -  
 190 Eqs.(4,5) - and, consequently, on both the temporal coherent and incoherent  
 191 components of the radiation energy spectrum, Eq.(6).

192 In conclusion, in the most general case of an electron bunch at a normal  
 193 angle of incidence onto a radiator surface  $S$  with an arbitrary size and shape  
 194 (finite  $S < \infty$  or infinite  $S = \infty$ ), Eq.(6) represents a temporal causal and  
 195 covariance consistent formulation of the transition radiation energy spectrum.

### 196 **3. Covariance of the charge form factor in the transition radiation** 197 **energy spectrum**

#### 198 *3.1. Covariance of the electron bunch density*

199 The covariance of the charge form factor and, in general, of the transition  
 200 radiation energy spectrum - as formulated in Eqs.(4,5,6) - will be analyzed  
 201 in the following paragraph. For such a purpose, the Fourier transforma-  
 202 tions of fields and charge distributions as normally defined in the ordinary  
 203 3-dimensional space will be reformulated in an explicitly covariant form as  
 204 4-dimensional Fourier representations in the conjugate Fourier wavevector-  
 205 frequency 4-space  $(k_x, k_y, k_z, \omega/c)$  of the space-time  $(x, y, z, ct)$  in the labo-  
 206 ratory reference frame  $\mathbb{R}$  or in the conjugate Fourier 4-space  $(k'_x, k'_y, k'_z, \omega'/c)$   
 207 of the space-time 4-vector  $(x', y', z', ct')$  in the rest reference frame  $\mathbb{R}'$ , see  
 208 Appendix A, Appendix B, Appendix C and Appendix D. The case of the  
 209 covariant transformation of the spatial density of a  $N$  electron bunch, as in  
 210 the following described, will exemplify the above mentioned explicitly covari-  
 211 ant extension of a Fourier representation of a field or a charge distribution  
 212 from the ordinary 3-dimensional space to the Lorentz 4-space.

213 In the present context, the  $N$  electrons of the bunch are described by a  
 214 “frozen” in time spatial distribution function: all the electrons are supposed  
 215 to move with a rectilinear and uniform velocity  $\vec{w} = (0, 0, w)$  along the  $z$  axis  
 216 of the laboratory reference frame  $\mathbb{R}$ . In the reference frame  $\mathbb{R}'$ , where the  $N$   
 217 electrons are at rest, the distribution function of the spatial density of the  
 218 electron bunch reads:

$$\rho'(\vec{r}', t') = \rho'(\vec{r}', 0) = \sum_{j=1}^N \delta(\vec{r}' - \vec{r}'_j), \quad (8)$$

219 where  $\vec{r}'_j = (x'_j, y'_j, z'_j)$  ( $j = 1, \dots, N$ ) are the spatial coordinates of the  $N$   
 220 electrons at the time  $t'$  in  $\mathbb{R}'$ . In the ordinary 3-space  $\vec{r}' = (x', y', z')$  and the



221 conjugate Fourier 3-space  $\vec{k}' = (k'_x, k'_y, k'_z)$  of  $\mathbb{R}'$ , the spatial density of the  $N$   
 222 electrons - Eq.(8) - reads

$$\rho'(\vec{r}', t') = \frac{1}{(2\pi)^3} \sum_{j=1}^N \int d\vec{k}' e^{i\vec{k}' \cdot (\vec{r}' - \vec{r}'_j)}. \quad (9)$$

223 The above Fourier transformation of the spatial density of the  $N$  electron  
 224 bunch is defined in the space  $(x', y', z')$  and in the conjugate Fourier space  
 225  $(k'_x, k'_y, k'_z)$  of the rest frame of reference  $\mathbb{R}'$ , which are subspaces of the  
 226 space-time  $(x', y', z', ct')$  and of the wavevector-frequency  $(k'_x, k'_y, k'_z, \omega'/c)$ ,  
 227 respectively. Consequently, the charge density, as expressed by Eq.(9) in  
 228 the ordinary 3-dimensional space of  $\mathbb{R}'$ , cannot explicitly show the expected  
 229 covariance under a Lorentz transformation from the rest to the laboratory  
 230 reference frame ( $\mathbb{R}' \Rightarrow \mathbb{R}$ ). An explicitly covariant Fourier representation  
 231 of the spatial density of the  $N$  electrons can be obtained by extending the  
 232 Fourier transformation as given in Eq.(9) into the Fourier conjugate 4-space  
 233  $(k'_x, k'_y, k'_z, \omega'/c)$  of the reference frame  $\mathbb{R}'$ , for details see Appendix B. With  
 234 reference to Appendix B, the spatial density of the  $N$  electron bunch - as  
 235 given in Eq.(9) - can be represented in an explicitly covariant form in the  
 236 space-time and wavevector-frequency 4-spaces of the rest reference frame  $\mathbb{R}'$   
 237 as follows:

$$\rho'(\vec{r}', t') = \frac{c}{(2\pi)^4} \int d^4\kappa' e^{i\kappa' \cdot \xi'} \rho(\vec{k}', \omega') \quad (10)$$

238 where  $d^4\kappa' = d\vec{k}' d(\omega'/c)$  and

$$\rho(\vec{k}', \omega') = 2\pi \left( \sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j} \right) \delta(\omega'), \quad (11)$$

239 where  $\xi'_j = (x'_j, y'_j, z'_j, ct'_j)$  ( $j = 1, \dots, N$ ) are the space-time 4-vectors of the  
 240 electron coordinates, while  $\xi' = (x', y', z', ct')$  and  $\kappa' = (k'_x, k'_y, k'_z, \omega'/c)$  are  
 241 the space-time and the conjugate Fourier wavevector-frequency 4-vectors in  
 242 the rest reference frame  $\mathbb{R}'$ , respectively.

243 The covariance of the Fourier representation of the spatial density of  $N$   
 244 electron bunch - as represented in Eq.(9) or, equivalently, in Eqs.(10,11) - can  
 245 be now easily checked, see also Appendix B. With reference to Eqs.(10,11),  
 246  $\rho(\vec{k}', \omega')$  is the only non-Lorentz-invariant quantity in the integrand of Eq.(10).

Moreover, according to Eqs.(11), the covariance of  $\rho(\vec{k}', \omega')$  only depends on how the delta Dirac function  $\delta(\omega')$  transforms under a Lorentz transformation. Under a Lorentz transformation from the rest to the laboratory reference frame ( $\mathbb{R}' \implies \mathbb{R}$ ) - see Appendix A and Appendix B - the delta Dirac function transforms indeed as

$$\delta(\omega') = \delta[\gamma(\omega - w k_z)] = \frac{\delta(\omega - \vec{w} \cdot \vec{k})}{\gamma}. \quad (12)$$

In conclusion, under a Lorentz transformation ( $\mathbb{R}' \implies \mathbb{R}$ ), the  $N$  electron spatial density - as represented by Eq.(9) in the ordinary 3-space or, equivalently, by Eqs.(10,11) in the Lorentz 4-space - transforms in agreement with the expected time-like covariance as

$$\rho'(\vec{k}', \omega') = \frac{1}{\gamma} \rho(\vec{k}, \omega), \quad (13)$$

where

$$\rho(\vec{k}, \omega) = 2\pi \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \delta(\omega - \vec{w} \cdot \vec{k}) \quad (14)$$

is the 4-dimensional Fourier transformation of the  $N$  electron spatial density in the laboratory reference frame  $\mathbb{R}$ , see also Appendix B.

### 3.2. Covariance of the electron bunch form factor

In the present context, a bunch of  $N$  electrons moving with a common rectilinear and uniform velocity  $\vec{w} = (0, 0, w)$  along the  $z$ -axis of the laboratory reference frame is supposed to strike, at a normal angle of incidence, a flat metallic screen placed in the plane  $z = 0$ . For such an experimental situation, Equations (4,5) and (6) formulate the transition radiation energy spectrum in the most general case of a radiator surface  $S$  having an arbitrary shape and size (either finite  $S < \infty$  or infinite  $S = \infty$ ).

This formulation of the radiation energy spectrum is very general. It is indeed expressed in terms of the original discrete distribution function of the  $N$  electron coordinates: no continuous limit approximation is indeed applied to the distribution function of the electron spatial coordinates. Furthermore, the size and the shape of the radiator surface being indeed arbitrary,

272 Eqs.(4,5,6) represent a very general and implicit formulation of the transi-  
 273 tion radiation energy spectrum of a  $N$  electron bunch which is invariant in  
 274 relation to the size and shape of the radiator.

275 This formulation of the radiation energy spectrum also meets the tem-  
 276 poral causality constraint. For the electromagnetic radiative mechanism of  
 277 the  $N$  electron bunch that is described in the present paper, the causality  
 278 constraint rules the temporal correlation between the single electron colli-  
 279 sion onto the metallic screen and the consequent emission from the metallic  
 280 surface of the corresponding single electron radiation field amplitude  $H_{\mu,j}$   
 281 ( $j = 1, \dots, N$ ), see Eqs.(4,5). In the considered experimental context, the  
 282 temporal sequence of the  $N$  electron collisions onto the metallic screen is  
 283 only ruled by the distribution function of the longitudinal coordinates  $z_{0j}$   
 284 ( $j = 1, \dots, N$ ) of the  $N$  electrons. Thanks to the implicit integral formulation  
 285 of the  $N$  single electron amplitudes  $H_{\mu,j}$  ( $j = 1, \dots, N$ ) composing the radi-  
 286 ation field - see Eqs.(4,5) - the general formula of the transition radiation  
 287 energy spectrum - Eq.(6) - explicitly meet the temporal causality constraint.  
 288 In Eq.(6) indeed, the dependence of the  $N$  single electron field amplitudes  
 289 on the electron transverse coordinates  $\vec{\rho}_{0j} = (x_{0j}, y_{0j})$  ( $j = 1, \dots, N$ ) being en-  
 290 coded in the implicit integral formulation given in Eqs.(4,5), the interference  
 291 between the  $N$  single electron field amplitudes  $H_{\mu,j}$  ( $j = 1, \dots, N$ ) is a di-  
 292 rect function of the corresponding emission phases from the metallic surface,  
 293  $e^{-i(\omega/w)(z_{0j}-z_{0l})}$  ( $l \neq j$ ), in agreement with the temporal causal principle.

294 About the covariance of the transition radiation energy spectrum of the  $N$   
 295 electron bunch as formulated in Eqs.(4,5,6), this can be argued on the basis of  
 296 the following analysis: first, verify the covariance of the formal steps leading  
 297 from the *virtual quanta* field of the  $N$  electron bunch to the resultant radiation  
 298 energy spectrum, see Eq.(3) and Eqs.(4,5,6); finally, verify the covariance of  
 299 the given expression of the *virtual quanta* field under a Lorentz transformation  
 300 from the laboratory reference frame  $\mathbb{R}$  to the reference frame  $\mathbb{R}'$  of rest of the  
 301  $N$  electrons.

302 About the covariance of the formal procedure leading from the *virtual*  
 303 *quanta* field to the radiation field, the following observation can be done. On  
 304 the basis of the far-field implementation of the Helmholtz-Kirchhoff integral  
 305 theorem [7, 57, 58], a given harmonic component of the transition radiation  
 306 field of the  $N$  electron bunch can be obtained by Fourier transforming the  
 307 corresponding *virtual quanta* field with respect to the spatial coordinates of  
 308 the radiator surface  $S$ , see Eq.(2). The calculation of such a Fourier trans-  
 309 formation only involves the coordinates  $(x, y)$  of the radiator surface  $S$  and

the related conjugate Fourier coordinates  $(k_x, k_y)$  which are transverse with respect to the direction of motion of the electron bunch and thus invariant under a Lorentz transformation. Therefore, the covariance properties of the *virtual quanta* field are entirely and unalterably transferred into the transition radiation field - see Eq.(3) and Eq.(4) - via the Fourier transformation as defined in Eq.(2).

Finally, about the covariance of the charge form factor formulation in the transition radiation energy spectrum - Eqs.(4,5,6) - this can be verified by analyzing how the *virtual quanta* field of the  $N$  electron bunch transforms under a Lorentz transformation from the laboratory to the rest reference frame ( $\mathbb{R} \implies \mathbb{R}'$ ), see also Appendix A.

In the present context, a “frozen-in-time” electron bunch is considered, i.e., the  $N$  electrons in motion with the same velocity  $\vec{w} = (0, 0, w)$  are described by a distribution function of the spatial coordinates that is invariant under a time-space translation in the laboratory reference frame  $\mathbb{R}$ , see also Appendix B

$$\begin{cases} \vec{r} = \vec{r}_0 + \vec{w}t \\ \vec{r}_j = \vec{r}_{0j} + \vec{w}t \end{cases}$$

where  $\vec{r}_{0j}$  ( $j = 1, \dots, N$ ) are the  $N$  electron spatial coordinates at the time  $t = 0$  when, for instance, the center of mass of the  $N$  electron coordinates is supposed to cross the plane  $z = 0$ .

For such an experimental situation, with reference to [60, 61, 62, 63, 64, 65], the expression of the charge electric field reads, see also Appendix D:

$$\vec{E}(\vec{k}, \omega) = -i(8\pi^2 e) \frac{[\vec{k} - (\omega \vec{w}/c^2)]}{[k^2 - (\omega/c)^2]} \left( \sum_{j=1}^N e^{-i\vec{k} \cdot \vec{r}_{0j}} \right) \delta(\omega - \vec{w} \cdot \vec{k}). \quad (15)$$

The expression above of the electric field of the  $N$  electron bunch does not explicitly show the expected covariance as the argument of the phase factor in Eq.(15) clearly indicates. Following the procedure already described in previous subsection 3.1 and in Appendix B, Appendix C and Appendix D, the explicit covariance of the electric field of the  $N$  electrons can be retrieved by extending the Fourier representation of the charge electric field from the ordinary 3-dimensional spaces of the spatial coordinates  $\vec{r} = (x, y, z)$  and conjugate wave-vector coordinates  $\vec{k} = (k_x, k_y, k_z)$  into the Lorentz 4-space of the space-time  $\xi = (\vec{r}, ct)$  and the conjugate Fourier 4-space of the wavevector-frequency  $\kappa = (\vec{k}, \omega/c)$ . With reference to the time-space translation of the

341 spatial coordinates of the  $N$  electrons in the laboratory reference frame  $\mathbb{R}$ ,  
 342 see Equations above, the Fourier representation of the electric field of the  $N$   
 343 electrons - given in Eq.(15) - can be reformulated in the following explicitly  
 344 covariant form, see also Eqs.(D.1,D.2) in Appendix D:

$$\vec{E}(\vec{r}, t) = \frac{1}{(2\pi)^4} \int d\vec{k} d\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} \vec{E}(\vec{k}, \omega) = \frac{c}{(2\pi)^4} \int d^4\kappa e^{i\kappa \cdot \xi} \vec{E}(\vec{k}, \omega) \quad (16)$$

345 where  $d^4\kappa = d\vec{k} d(\omega/c)$  and, taking into account the following time-space  
 346 translation  $\vec{r}_j = \vec{r}_{0j} + \vec{w}t$  (see coordinates transformation above),

$$\begin{aligned} \vec{E}(\vec{k}, \omega) &= -i(8\pi^2 e) \frac{[\vec{k} - (\omega \vec{w}/c^2)]}{[k^2 - (\omega/c)^2]} \left( \sum_{j=1}^N e^{-i\vec{k} \cdot (\vec{r}_j - \vec{w}t)} \right) \delta(\omega - \vec{w} \cdot \vec{k}) \\ &= -i(8\pi^2 e) \frac{[\vec{k} - (\omega \vec{w}/c^2)]}{\kappa^2} \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \delta(\omega - \vec{w} \cdot \vec{k}), \end{aligned} \quad (17)$$

347 where the 4-vector  $\xi_j = (\vec{r}_j, ct)$  ( $j = 1, \dots, N$ ) represents the space-time coor-  
 348 dinates of the  $N$  electrons in the laboratory reference frame.

349 The covariance of the electric field of the  $N$  electron bunch - see Eq.(17)  
 350 or equivalently Eq.(15) - can be finally verified, as in following argued. In  
 351 fact, looking at the integrand of Eq.(16) and at Eq.(17), the term  $[k_z -$   
 352  $(\omega w/c^2)]\delta(\omega - \vec{w} \cdot \vec{k})$  is the only non-Lorentz-invariant quantity whose covari-  
 353 ance has to be checked. Under a Lorentz transformation from the laboratory  
 354 to the rest reference frame ( $\mathbb{R} \Rightarrow \mathbb{R}'$ ), such a quantity transforms indeed as,  
 355 see also Appendix A and Appendix D:

$$\begin{cases} \delta(\omega - \vec{w} \cdot \vec{k}) \longrightarrow \delta[\gamma(\omega' + w k'_z) - w\gamma(k'_z + \beta\omega'/c)] = \delta(\frac{\omega'}{\gamma}) = \gamma\delta(\omega') \\ (k_z - \omega w/c^2) \longrightarrow \gamma(k'_z + \beta\omega'/c) - \gamma(\omega' + \beta c k'_z)w/c^2 = \frac{k'_z}{\gamma} \end{cases}$$

356 Finally, under a Lorentz transformation  $\mathbb{R} \Rightarrow \mathbb{R}'$ , Eqs.(16,17) transform as:

$$\begin{aligned} \vec{E}(\vec{r}, t) &= -\frac{iec}{2\pi^2} \int d^4\kappa e^{i\kappa \cdot \xi} \frac{\sum_{j=1}^N e^{-i\kappa \cdot \xi_j}}{\kappa^2} [\vec{k} - (\omega \vec{w}/c^2)] \delta(\omega - \vec{w} \cdot \vec{k}) = \\ &= -\frac{iec}{2\pi^2} \int d^4\kappa' e^{i\kappa' \cdot \xi'} \frac{\sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j}}{\kappa'^2} \begin{pmatrix} \gamma k'_x \\ \gamma k'_y \\ k'_z \end{pmatrix} \delta(\omega') = \\ &= \begin{pmatrix} \gamma E'_x(\vec{r}', t') \\ \gamma E'_y(\vec{r}', t') \\ E'_z(\vec{r}', t') \end{pmatrix}, \end{aligned} \quad (18)$$

where, in the last term of Eq.(18), the three components of the charge electric field  $\vec{E}'(\vec{r}', t')$  in the rest reference frame  $\mathbb{R}'$  can be identified, see also Appendix C:

$$\vec{E}'(\vec{r}', t') = -\frac{iec}{2\pi^2} \int d^4\kappa' e^{i\kappa' \cdot \xi'} \frac{\sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j}}{\kappa'^2} \vec{k}' \delta(\omega'). \quad (19)$$

Equation(19) is the explicitly covariant Fourier representation of the electric field of a  $N$  electron bunch in the rest reference frame  $\mathbb{R}'$ , where  $(x', y', z', ct')$  and  $(k'_x, k'_y, k'_z, \omega'/c)$  are, respectively, the space-time and the conjugate Fourier wavevector-frequency 4-vectors of  $\mathbb{R}'$  and  $\xi'_j = (\vec{r}'_j, ct')$  ( $j = 1, \dots, N$ ) is the space-time 4-vectors representing the  $N$  electron coordinates in  $\mathbb{R}'$ , see also Appendix C and Appendix D.

In conclusion, the expression of the *virtual quanta* field, see Eq.(17) or equivalently Eq.(15), which in the present and other works [60, 61, 62, 63, 64, 65] is the starting point to the formulation of the radiation energy spectrum of a  $N$  electron bunch, is demonstrated to meet the covariant constraint. The formulation of the radiation energy spectrum - see Eqs.(4,5,6) - is also demonstrated to be covariant and temporal causality consistent. Equations (4,5,6) are indeed the final result of a series of temporal-causality-consistent formal steps which preserve and transmit the original covariance of the *virtual quanta* field of the  $N$  electron beam into the radiation energy spectrum.

Finally, about the invariance of the transverse projection of the  $N$  electron density with respect to the direction of motion and about the covariant imprinting of it on the charge electric field and, consequently, on the radiation field and on the radiation energy spectrum, the following observation can be done. The effect of the transverse electron coordinates  $\vec{\rho}_{0j} = (x_{0j}, y_{0j})$  ( $j = 1, \dots, N$ ) on the Fourier transformation of the charge electric field is driven by a phase factor, see Eqs.(16,17) and Eqs.(18,19). This is a function of the Lorentz invariant quantity  $k_x x_{0j} + k_y y_{0j}$  ( $j = 1, \dots, N$ ), where  $(k_x, k_y)$  are the transverse component of the wave-vector. With reference to Eqs.(16,17) and Eqs.(18,19), the charge electric field shows a dependence on the transverse electron coordinates which, under a Lorentz transformation from the laboratory to the rest reference frame, transfers and transforms in a covariant way and whose observability at a given wavelength remains unchanged, this being determined by the Lorentz-invariant scalar product of the electron transverse coordinates and the transverse components of the wave-vector. In the light of such a meaning, the statement that, see also end of section 2,

the Lorentz invariance of the transverse projection of the  $N$  electron density is expected to leave a covariant mark on the  $N$  electron radiation field amplitudes - Eqs.(4,5) - and, consequently, on both the temporal coherent and incoherent components of the radiation energy spectrum - see Eq.(6) - has to be read and integrated with the statement that the observability of the covariant effect of the transverse coordinates on the charge electric field remains unchanged, passing from an inertial reference frame to the other, this being a function of a Lorentz-invariant quantity.

#### 4. Conclusions

In the present paper, the issue of the covariant formulation of the radiation energy spectrum and, consequently, of the charge form factor of a  $N$  electron bunch is argued. The  $N$  electron bunch is supposed to collide, at a normal angle of incidence, onto a metallic screen  $S$  which has an arbitrary shape and size (either finite  $S < \infty$  or infinite  $S = \infty$ ) and is supposed to behave as an ideal conductor in the wavelength region of interest. As the covariance of a charged distribution is expected to evolve from a charge-point-like into a charge-density-like one when passing from a single electron to a  $N$  electron bunch, the covariance of the radiation energy spectrum as well is expected to behave in the same way when a  $N$  electron bunch is considered instead of a single electron. In order to verify the covariant formulation of the radiation energy spectrum, the formal steps leading from the virtual quanta field to the transition radiation energy spectrum of a  $N$  electron bunch have been explicitly derived and checked to meet, first, the temporal causality constraint and, then, to be covariance consistent. Finally, the expression of the *virtual quanta* field, which, in the present paper and in other papers [60, 61, 62, 63, 64, 65], is the starting point to achieve the formula of the radiation energy spectrum, has been checked to transform in a covariant way under a Lorentz transformation from the laboratory reference frame ( $\mathbb{R}$ ) to the reference frame of rest of the  $N$  electron bunch ( $\mathbb{R}'$ ). In order to perform such a covariance check, the Fourier representations of fields and charged distributions have been suitably extended from the ordinary 3-dimensional space of the spatial coordinates and the conjugate Fourier wave-vectors into the Lorentz 4-spaces of the space-time and of the conjugate Fourier wavevector-frequency in both  $\mathbb{R}$  and  $\mathbb{R}'$ . In a temporal causal and covariant formulation of the transition radiation energy spectrum of a  $N$  electron bunch, the invariance of the projection of the electron density in

the transverse plane with respect to the direction of motion manifests itself as a covariant feature of the  $N$  single electron amplitude composing the radiation field and, consequently, of both the temporal coherent and incoherent components of the transition radiation energy spectrum.

## Appendix A. Lorentz transformations of coordinates and fields

In the present work, the case of a bunch of  $N$  electrons in a rectilinear and uniform motion with a common velocity  $\vec{w} = (0, 0, w)$  along the positive direction of the  $z$ -axis of the laboratory reference frame  $\mathbb{R}$  is considered. In the present paper, Lorentz transformations from the laboratory reference frame  $\mathbb{R}$  to the reference frame of rest  $\mathbb{R}'$  of the  $N$  electron bunch - and viceversa - are applied to fields and spatial distributions of charges. Therefore, for the sake of easy reading, the main results are below reported.

The 4-vectors of  $\mathbb{R}$ , the space-time  $(x, y, z, ct)$  and the conjugate Fourier wavevector-frequency  $(k_x, k_y, k_z, \omega/c)$ , Lorentz-transform into the respective counterparts  $(x', y', z', ct')$  and  $(k'_x, k'_y, k'_z, \omega'/c)$  of  $\mathbb{R}'$  according to [57] ( $\mathbb{R}' \Rightarrow \mathbb{R}$ ):

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma(z - \beta x_0) \\ x'_0 = \gamma(x_0 - \beta z) \end{cases}$$

$$\begin{cases} k'_x = k_x \\ k'_y = k_y \\ k'_z = \gamma(k_z - \beta k_0) \\ k'_0 = \gamma(k_0 - \beta k_z) \end{cases}$$

where  $\beta = w/c$  and  $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ .

As for the electromagnetic fields  $\vec{E}$  and  $\vec{B}$ , Lorentz transformations  $\mathbb{R}' \Rightarrow \mathbb{R}$  read:

$$\begin{cases} E'_x(\vec{r}', t') = \gamma(E_x(\vec{r}, t) - \beta B_y(\vec{r}, t)) \\ E'_y(\vec{r}', t') = \gamma(E_y(\vec{r}, t) + \beta B_x(\vec{r}, t)) \\ E'_z(\vec{r}', t') = E_z(\vec{r}, t) \end{cases}$$

and

$$\begin{cases} B'_x(\vec{r}', t') = \gamma(B_x(\vec{r}, t) + \beta E_y(\vec{r}, t)) \\ B'_y(\vec{r}', t') = \gamma(B_y(\vec{r}, t) - \beta E_x(\vec{r}, t)) \\ B'_z(\vec{r}', t') = B_z(\vec{r}, t) \end{cases}$$



448 The inverse Lorentz transformations  $\mathbb{R} \implies \mathbb{R}'$  for coordinates and fields  
 449 can be obtained from previous equations by substituting  $\beta \rightarrow -\beta$  and by  
 450 exchanging the prime in field components and coordinates.

## 451 **Appendix B. Covariance of the spatial density of a bunch of $N$** 452 **electrons**

453  $N$  electrons in a bunch are supposed to move with a common rectilinear  
 454 and uniform velocity in the laboratory reference frame  $\mathbb{R}$ . In the frame of  
 455 reference  $\mathbb{R}'$  where each electron is at rest, the  $N$  electrons can be described  
 456 in terms of a “static” distribution function of the particle density:

$$\rho'(\vec{r}', t') = \rho'(\vec{r}', 0) = \sum_{j=1}^N \delta(\vec{r}' - \vec{r}'_j) \quad (\text{B.1})$$

457 where  $\vec{r}' = (x', y', z')$  and  $\vec{r}'_j = (x'_j, y'_j, z'_j)$  ( $j = 1, \dots, N$ ) are the  $N$  electron  
 458 coordinates in the rest reference frame  $\mathbb{R}'$ . In the conjugate Fourier space  
 459  $(k'_x, k'_y, k'_z)$  of the ordinary 3-space of the spatial coordinates  $(x', y', z')$  of  $\mathbb{R}'$ ,  
 460 the equation above reads

$$\rho'(\vec{r}', t') = \frac{1}{(2\pi)^3} \int d\vec{k}' e^{i\vec{k}' \cdot \vec{r}'} \rho'(\vec{k}') \quad (\text{B.2})$$

461 where

$$\rho'(\vec{k}') = \sum_{j=1}^N e^{-i\vec{k}' \cdot \vec{r}'_j} = \sum_{j=1}^N e^{-i(k'_x x'_j + k'_y y'_j + k'_z z'_j)} \quad (\text{B.3})$$

462 is the Fourier transform of the distribution function of the  $N$  particle density.  
 463 Since the bunch density - as represented by Eq.(B.2) - is only defined in a  
 464 subspace  $(k'_x, k'_y, k'_z)$  of the wavevector-frequency 4-vector  $(k'_x, k'_y, k'_z, \omega'/c)$  of  
 465  $\mathbb{R}'$ , it cannot explicitly show the expected Lorentz-covariance. The explicit  
 466 covariance of the spatial density of the  $N$  electrons can be directly retrieved  
 467 by extending the Fourier representation of Eqs.(B.2,B.3) into the wavevector-  
 468 frequency  $(k'_x, k'_y, k'_z, \omega'/c)$  and space-time  $(x', y', z', ct')$  4-spaces of the rest  
 469 reference frame  $\mathbb{R}'$ . Such a result can be achieved by extending the integral  
 470 of Eq.(B.2) into the 4-space  $\kappa = (k'_x, k'_y, k'_z, \omega'/c)$  and the integrand into itself  
 471 times  $\delta(\omega') e^{i\omega' t'} e^{-i\omega' t'}$  where  $\delta(\omega')$  is a delta Dirac function. Following such

472 a formal procedure, Eq.(B.2) can be reformulated as follows:

$$\begin{aligned}
\rho'(r', t') &= \frac{c}{(2\pi)^3} \int d\vec{k}' d(\omega'/c) e^{i(\vec{k}' \cdot \vec{r}' - \omega' t')} \left( \sum_{j=1}^N e^{-i(\vec{k}' \cdot \vec{r}'_j - \omega' t')} \right) \delta(\omega') = \\
&= \frac{c}{(2\pi)^3} \int d^4 \kappa' e^{i\kappa' \cdot \xi'} \left( \sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j} \right) \delta(\omega') = \\
&= \frac{c}{(2\pi)^4} \int d^4 \kappa' e^{i\kappa' \cdot \xi'} \rho(\vec{k}', \omega')
\end{aligned} \tag{B.4}$$

473 where

$$\rho'(\vec{k}', \omega') = 2\pi \left( \sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j} \right) \delta(\omega') \tag{B.5}$$

474 with  $\kappa' = (k'_x, k'_y, k'_z, \omega'/c)$  and  $\xi' = (x', y', z', ct')$  and where  $\xi'_j = (x'_j, y'_j, z'_j, ct')$   
475 ( $j = 1, \dots, N$ ) are the space-time coordinate 4-vectors of the electrons in the  
476 rest reference frame  $\mathbb{R}'$ . Thanks to the extension of the Fourier represen-  
477 tation of the spatial density of the  $N$  electron bunch into the 4-space of  
478 the rest reference frame  $\mathbb{R}'$  - see Eqs.(B.4,B.5) - the expected covariance of  
479 the charge density can be explicitly recovered. The following quantities are  
480 indeed Lorentz-invariant

$$\begin{cases} d^4 \kappa' = d\vec{k}' d(\omega'/c) = d^4 \kappa \\ \kappa' \cdot \xi' = k'_x x' + k'_y y' + k'_z z' - \omega' t' = \kappa \cdot \xi \\ \kappa' \cdot \xi'_j = k'_x x'_j + k'_y y'_j + k'_z z'_j - \omega' t' = \kappa \cdot \xi_j \end{cases}$$

481 Consequently, under a Lorentz transformation from the rest to the labora-  
482 tory reference frame ( $\mathbb{R}' \Rightarrow \mathbb{R}$ ), Eq.(B.5) transforms as follows, see also  
483 Appendix A:

$$\begin{aligned}
\rho'(\vec{k}', \omega') &= 2\pi \left( \sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j} \right) \delta(\omega') = 2\pi \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \delta[\gamma(\omega - w k_z)] = \\
&= 2\pi \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \frac{\delta(\omega - \vec{w} \cdot \vec{k})}{\gamma} = \frac{1}{\gamma} \rho(\vec{k}, \omega),
\end{aligned} \tag{B.6}$$

484 where

$$\rho(\vec{k}, \omega) = 2\pi \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \delta(\omega - \vec{w} \cdot \vec{k}) \tag{B.7}$$

485 represents the 4-dimensional Fourier transformation of the distribution func-  
 486 tion of the spatial density of the  $N$  electron bunch in the laboratory reference  
 487 frame  $\mathbb{R}$ . In the equation above,  $\xi_j = (x_j, y_j, z_j, ct)$  ( $j = 1, \dots, N$ ) are the 4-  
 488 vector space-time coordinates of the electrons in  $\mathbb{R}$  as well as  $\xi = (x, y, z, ct)$   
 489 and  $\kappa = (k_x, k_y, k_z, \omega/c)$  are, respectively, the space-time and the conjugate  
 490 Fourier wavevector-frequency Lorentz 4-vectors of  $\mathbb{R}$ .

491 In conclusion, under a Lorentz transformation  $\mathbb{R}' \implies \mathbb{R}$ , the distribution  
 492 function of the spatial density of the  $N$  electrons - see Eqs.(B.1,B.2) and  
 493 Eqs.(B.6,B.7) - transforms according to the expected covariance:

$$\rho'(\vec{r}', t') = \frac{1}{\gamma} \rho(\vec{r}, t), \quad (\text{B.8})$$

494 where

$$\begin{aligned} \rho(\vec{r}, t) &= \frac{c}{(2\pi)^3} \int d^4\kappa e^{i\kappa \cdot \xi} \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \delta(\omega - \vec{w} \cdot \vec{k}) = \\ &= \frac{1}{(2\pi)^3} \int d\vec{k} e^{i\vec{k} \cdot (\vec{r} - \vec{w}t)} \left( \sum_{j=1}^N e^{-i\vec{k} \cdot (\vec{r}_j - \vec{w}t)} \right) = \rho(\vec{r} - \vec{w}t) = \\ &= \frac{1}{(2\pi)^3} \int d\vec{k} e^{i\vec{k} \cdot \vec{r}_0} \left( \sum_{j=1}^N e^{-i\vec{k} \cdot \vec{r}_{0j}} \right) = \sum_{j=1}^N \delta(\vec{r}_0 - \vec{r}_{0j}) = \rho(\vec{r}_0, \text{B.9}) \end{aligned}$$

495 represents the distribution function of the  $N$  electron density in the labora-  
 496 tory reference frame  $\mathbb{R}$ , see also Eqs.(5,6) in [61].

497 The distribution function of the spatial density of the  $N$  electrons - as  
 498 represented by Eq.(B.9) - is invariant under a time-space translation in the  
 499 laboratory reference frame  $\mathbb{R}$

$$\begin{cases} \vec{r} = \vec{r}_0 + \vec{w}t \\ \vec{r}_j = \vec{r}_{0j} + \vec{w}t \end{cases}$$

500 or, in other words, it is “frozen-in-time” as already mentioned in the present  
 501 paper where the case of a  $N$  electron bunch in a rectilinear and uniform  
 502 motion is considered. Looking at both integrand of Eqs.(B.4,B.5,B.7,B.9),  
 503 the presence of a delta Dirac function and, in particular, the argument of  
 504 this delta are instructive about the attribute given to the charge distribution  
 505 function to be “static” in the rest reference frame  $\mathbb{R}'$  or “frozen-in-time” in

the laboratory reference frame  $\mathbb{R}$ . In the reference frame  $\mathbb{R}'$ , where the  $N$  electrons are at rest,  $\delta(\omega')$  in Eq.(B.5) is indicating that only a wave with  $\omega' = 0$  constitutes the relevant harmonic contribution to the Fourier transformation of the charge density. Conversely, in the laboratory reference frame  $\mathbb{R}$ , where all the  $N$  electrons move with a common rectilinear and uniform velocity  $\vec{w}$ , the delta Dirac  $\delta(\omega - \vec{w} \cdot \vec{k})$  in Eq.(B.9) is indicating that a plane wave traveling with the same velocity  $\vec{w}$  as the electron bunch constitutes the only relevant harmonic contribution to the Fourier transformation of the charge density.

### Appendix C. Electric field of a $N$ electron bunch in the rest reference frame

In the rest reference frame  $\mathbb{R}'$  of the  $N$  electron bunch, the magnetic field  $\vec{B}'(\vec{r}', t') \equiv 0$ . The electric field  $\vec{E}'(\vec{r}') = -\vec{\nabla}\Phi'(\vec{r}')$  can be obtained from the Poisson equation for the scalar potential  $\Phi$ :

$$-\nabla^2\Phi'(\vec{r}') = 4\pi e \rho'(\vec{r}'). \quad (\text{C.1})$$

In the Fourier 3-space  $\vec{k}' = (k'_x, k'_y, k'_z)$  of  $\mathbb{R}'$ , where the scalar potential and the charge density read

$$\begin{cases} \Phi'(\vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k}' e^{i\vec{k}' \cdot \vec{r}'} \Phi'(\vec{k}') \\ \rho'(\vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k}' e^{i\vec{k}' \cdot \vec{r}'} \rho'(\vec{k}') \end{cases}$$

the Poisson equation reads

$$\Phi'(\vec{k}') = 4\pi e \frac{\rho'(\vec{k}')}{k'^2}. \quad (\text{C.2})$$

and the electric field reads

$$\begin{aligned} \vec{E}'(\vec{r}') &= -\vec{\nabla}\Phi'(\vec{r}') = -\frac{i e}{2\pi^2} \int d\vec{k}' e^{i\vec{k}' \cdot \vec{r}'} \frac{\vec{k}'}{k'^2} \rho'(\vec{k}') = \\ &= -\frac{i e}{2\pi^2} \int d\vec{k}' e^{i\vec{k}' \cdot \vec{r}'} \frac{\vec{k}'}{(k_x'^2 + k_y'^2 + k_z'^2)} \left( \sum_{j=1}^N e^{-i\vec{k}' \cdot \vec{r}_j'} \right). \end{aligned} \quad (\text{C.3})$$

where the explicit expression of  $\rho'(\vec{k}')$  is given in Eq.(B.3).

525 The Fourier representation of the electric field of a  $N$  electron bunch in the  
 526 rest reference frame  $\mathbb{R}'$  - as given in Eq.(C.3) - is evidently non-explicitly co-  
 527 variant being restricted to the subspaces  $\vec{k}' = (k'_x, k'_y, k'_z)$  and  $\vec{r}' = (r'_x, r'_y, r'_z)$ ,  
 528 respectively, of the wavevector-frequency  $\vec{\kappa}' = (k'_x, k'_y, k'_z, \omega'/c)$  and space-  
 529 time  $\vec{\xi}' = (x', y', z', ct')$  Lorentz 4-vectors of  $\mathbb{R}'$ . The covariance of the electric  
 530 field can be explicitly retrieved by suitably extending the Fourier transfor-  
 531 mation in Eq.(C.3) into the Fourier 4-space  $(k'_x, k'_y, k'_z, \omega'/c)$ . Following the  
 532 same formal procedure already described in Appendix B for the charge den-  
 533 sity, the Fourier transformation of the electric field of the  $N$  electron bunch  
 534 in the rest reference frame  $\mathbb{R}'$  - Eq.(C.3) - can be reformulated as:

$$\begin{aligned}
 \vec{E}'(\vec{r}') &= -\frac{ie}{2\pi^2} \int d\vec{k}' e^{i\vec{k}' \cdot \vec{r}'} \frac{\vec{k}'}{(k'^2_x + k'^2_y + k'^2_z)} \left( \sum_{j=1}^N e^{-i\vec{k}' \cdot \vec{r}'_j} \right) = \\
 &= -\frac{ie}{2\pi^2} \int d\vec{k}' d\omega' e^{i(\vec{k}' \cdot \vec{r}' - \omega' t')} \left( \sum_{j=1}^N e^{-i(\vec{k}' \cdot \vec{r}'_j - \omega' t'_j)} \right) \frac{\vec{k}' \delta(\omega')}{[k'^2_x + k'^2_y + k'^2_z - (\omega'/c)^2]} = \\
 &= -\frac{iec}{2\pi^2} \int d^4\kappa' e^{i\kappa' \cdot \xi'} \frac{(\sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j})}{\kappa'^2} \vec{k}' \delta(\omega') = \\
 &= \frac{c}{(2\pi)^4} \int d^4\kappa' e^{i\kappa' \cdot \xi'} \vec{E}'(\vec{k}', \omega'), \tag{C.4}
 \end{aligned}$$

535 where  $\xi'_j = (\vec{r}'_j, ct'_j) = (x'_j, y'_j, z'_j, ct'_j)$  are the space-time coordinates of the  
 536  $N$  electrons ( $j = 1, \dots, N$ ) in  $\mathbb{R}'$ . In the integrand of the equation above, the  
 537 only non-Lorentz invariant quantity is the Fourier transform of the electric  
 538 field  $\vec{E}'(\vec{k}', \omega')$ :

$$\vec{E}'(\vec{k}', \omega') = -i(8\pi^2 e) \frac{(\sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j})}{\kappa'^2} \vec{k}' \delta(\omega'). \tag{C.5}$$

539 The term  $\vec{k}' \delta(\omega')$  is the only quantity in Eqs.(C.4,C.5) whose covariance has  
 540 to be checked. Under a Lorentz transformation from the rest to the labo-  
 541 ratory reference frame ( $\mathbb{R}' \implies \mathbb{R}$ ) - see also Appendix A - such a quantity  
 542 transforms as:

$$\begin{cases} k'_{x,y} \delta(\omega') \longrightarrow k_{x,y} \delta[\gamma(\omega - wk_z)] = \frac{k_{x,y}}{\gamma} \delta(\omega - wk_z) \\ k'_z \delta(\omega') \longrightarrow \gamma(k_z - \frac{w\omega}{c^2}) \delta[\gamma(\omega - wk_z)] = (k_z - \frac{w\omega}{c^2}) \delta(\omega - wk_z) \end{cases}$$

543 where  $\vec{w} = (0, 0, w)$  is the velocity of the electrons in the laboratory reference  
 544 frame  $\mathbb{R}$ . Finally, under a Lorentz transformation  $\mathbb{R}' \implies \mathbb{R}$ , the electric field

545 of a  $N$  electron bunch in the rest reference frame  $\mathbb{R}'$  - see Eqs.(C.3,C.4,C.5)  
 546 - transforms into

$$\begin{aligned}
 \vec{E}'(\vec{r}') &= \frac{c}{(2\pi)^4} \int d^4\kappa' e^{i\kappa' \cdot \xi'} \vec{E}'(\vec{k}', \omega') = \\
 &= -\frac{iec}{2\pi^2} \int d^4\kappa e^{i\kappa \cdot \xi} \frac{(\sum_{j=1}^N e^{-i\kappa \cdot \xi_j})}{\kappa^2} \begin{pmatrix} k_x/\gamma \\ k_y/\gamma \\ k_z - w\omega/c^2 \end{pmatrix} \delta(\omega - \vec{w} \cdot \vec{k}) = \\
 &= \begin{pmatrix} E_x(\vec{r}, t)/\gamma \\ E_y(\vec{r}, t)/\gamma \\ E_z(\vec{r}, t) \end{pmatrix} \quad (C.6)
 \end{aligned}$$

547 where, in the equation above, the 3 components of the electric field of the  
 548  $N$  electron bunch in the laboratory reference frame  $\mathbb{R}$  can be recognized, see  
 549 Eqs.(16,17) and also Appendix D:

$$\vec{E}(\vec{r}, t) = -\frac{iec}{2\pi^2} \int d^4\kappa e^{i\kappa \cdot \xi} \frac{(\vec{k} - \vec{w}\omega/c^2)}{\kappa^2} \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \delta(\omega - \vec{w} \cdot \vec{k}). \quad (C.7)$$

#### 550 **Appendix D. Electromagnetic field of a $N$ electron bunch in the** 551 **laboratory reference frame**

552 In the case of a bunch of  $N$  electrons in motion in vacuum with a common  
 553 uniform and rectilinear velocity  $\vec{w}$  in the laboratory reference frame  $\mathbb{R}$ , the  
 554 Fourier representation of the propagation equations of the 4-potential  $(\vec{A}, \Phi)$   
 555 in the 4-space  $(\vec{k}, \omega/c)$  of  $\mathbb{R}$  reads in the gauge of Lorentz as

$$\begin{cases} (-\frac{\omega^2}{c^2} + k^2) \vec{A}(\vec{k}, \omega) = \frac{4\pi e}{c} \vec{w} \rho(\vec{k}, \omega) \\ (-\frac{\omega^2}{c^2} + k^2) \Phi(\vec{k}, \omega) = 4\pi e \rho(\vec{k}, \omega) \end{cases}$$

556 where  $\vec{J}(\vec{k}, \omega) = e\rho(\vec{k}, \omega)\vec{w}$  is the Fourier transformation of the current den-  
 557 sity vector. With reference to the equations above, the Fourier transforma-  
 558 tion of the charge electric field reads:

$$\begin{aligned}
 \vec{E}(\vec{k}, \omega) &= -i\vec{k}\Phi(\vec{k}, \omega) + \frac{i\omega}{c} \vec{A}(\vec{k}, \omega) = \\
 &= -i4\pi e \frac{(\vec{k} - \omega\vec{w}/c^2)}{[k^2 - (\omega/c)^2]} \rho(\vec{k}, \omega). \quad (D.1)
 \end{aligned}$$

559 With reference to Eq.(B.7) in Appendix B, the Fourier representation of the  
 560 electric field of the  $N$  electron bunch - as given in Eq.(D.1) - can be expressed  
 561 in an explicitly covariant form in the space-time  $\xi = (\vec{r}, ct)$  and in the 4-space  
 562 of the conjugate Fourier wavevector-frequency  $\vec{\kappa} = (\vec{k}, \omega/c)$  of the laboratory  
 563 reference frame  $\mathbb{R}$ , see also Eqs.(16,17) and Eq.(C.7) in Appendix C:

$$\vec{E}(\vec{k}, \omega) = -i8\pi^2 e \frac{(\vec{k} - \omega \vec{w}/c^2)}{\kappa^2} \left( \sum_{j=1}^N e^{-i\kappa \cdot \xi_j} \right) \delta(\omega - \vec{w} \cdot \vec{k}), \quad (\text{D.2})$$

564 where  $\kappa^2 = k^2 - (\omega/c)^2$  and  $\xi_j = (\vec{r}_j, ct)$  ( $j = 1, \dots, N$ ) are the space-time  
 565 4-vectors of the  $N$  electron coordinates in the laboratory reference frame  $\mathbb{R}$ .

566 Under a Lorentz transformation from the laboratory to the rest reference  
 567 frame ( $\mathbb{R} \Rightarrow \mathbb{R}'$ ), the non-Lorentz-invariant quantities in Eq.(D.2) trans-  
 568 forms as follows:

$$\begin{cases} k_z - \omega w/c^2 = k'_z/\gamma \\ \delta(\omega - wk_z) = \gamma \delta(\omega') \end{cases}$$

569 see also Appendix A. Taking into account the equations above, in the lab-  
 570 oratory reference frame  $\mathbb{R}$ , the covariance of the 4-dimensional Fourier rep-  
 571 resentation of the electric field of the  $N$  electron bunch - Eq.(D.2) - can be  
 572 finally checked. Under a Lorentz transformation  $\mathbb{R} \Rightarrow \mathbb{R}'$ , Eq.(D.2) trans-  
 573 forms indeed as

$$\vec{E}(\vec{k}, \omega) = -i8\pi^2 e \frac{\left( \sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j} \right)}{\kappa'^2} \begin{pmatrix} \gamma k'_x \\ \gamma k'_y \\ k'_z \end{pmatrix} \delta(\omega') = \begin{pmatrix} \gamma E'_x(\vec{k}', \omega') \\ \gamma E'_y(\vec{k}', \omega') \\ E'_z(\vec{k}', \omega') \end{pmatrix} \quad (\text{D.3})$$

574 where, in the second term of previous equation, the explicitly covariant  
 575 Fourier representation of the charge electric field  $\vec{E}'(\vec{k}', \omega')$  in the rest refer-  
 576 ence frame  $\mathbb{R}'$  can be recognized, see Eq.(19) and also Eq.(C.5) in Appendix C.

577 For the sake of completeness, the electric field of  $N$  electrons in the rest  
 578 reference frame  $\mathbb{R}'$  - see second term of Eq.(D.3) or, equivalently, Eq.(19) or  
 579 Eq.(C.5) in Appendix C - will be in the following obtained by Lorentz trans-  
 580 forming the electric and magnetic fields in the laboratory frame of reference  
 581  $\mathbb{R}$ . In the laboratory reference frame  $\mathbb{R}$ , the Fourier transformation of the  
 582 magnetic field by a  $N$  electron bunch reads:

$$\vec{B}(\vec{k}, \omega) = i\vec{k} \times \vec{A}(\vec{k}, \omega) = \frac{i8\pi^2 e (\sum_{j=1}^N e^{-i\kappa \cdot \xi_j})}{c\kappa^2} \begin{pmatrix} wk_y \\ -wk_x \\ 0 \end{pmatrix} \delta(\omega - \vec{k} \cdot \vec{w}) \quad (\text{D.4})$$

583 The 4-dimensional Fourier representation of the transverse components of the  
 584 charge electric field in  $\mathbb{R}'$  can be obtained from the corresponding components  
 585 of the electromagnetic field in  $\mathbb{R}$  as (see also Appendix A):

$$\begin{aligned}
 E'_{(x,y)}(\vec{k}', \omega') &= \gamma[E_{(x,y)}(\vec{k}, \omega) \mp \beta B_{(y,x)}(\vec{k}, \omega)] = \\
 &= \gamma(-i8\pi^2 e) \frac{\left(\sum_{j=1}^N e^{-i\kappa \cdot \xi_j}\right)}{\kappa^2} \delta(\omega - \vec{w} \cdot \vec{k}) \left[ \begin{pmatrix} k_x \\ k_y \end{pmatrix} \pm \beta^2 \begin{pmatrix} -k_x \\ k_y \end{pmatrix} \right] = \\
 &= \frac{-i8\pi^2 e}{\gamma} \frac{\left(\sum_{j=1}^N e^{-i\kappa \cdot \xi_j}\right)}{\kappa^2} \delta(\omega - wk_z) \begin{pmatrix} k_x \\ k_y \end{pmatrix} = \\
 &= -i8\pi^2 e \frac{\left(\sum_{j=1}^N e^{-i\kappa' \cdot \xi'_j}\right)}{\kappa'^2} \delta(\omega') \begin{pmatrix} k'_x \\ k'_y \end{pmatrix} \tag{D.5}
 \end{aligned}$$

586 where in the last term of previous equation the transverse component of the  
 587 Fourier transform of the charge electric field in the rest reference frame  $\mathbb{R}'$   
 588 can be recognized (see Appendix C, Eq.(C.5) in particular). Concerning the  
 589 longitudinal component  $E_z(\vec{k}, \omega)$ , the corresponding covariant transformation  
 590 is trivial, see Eq.(D.3) or Eq.(C.5).

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